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Interacting electromagnetic waves in general relativity

J B Griffiths
Department of Mathematics, University of Technology, Loughborough, Leicestershire LE11 3TU, UK

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Abstract. In the general theory of relativity electromagnetic waves interact nonlinearly through the field equations. A region of space–time containing two null electromagnetic waves is considered, and a new class of exact solutions of the Einstein–Maxwell equations is given. The solutions describe two waves following shear-free null geodesics 'twisting' through each other with zero contraction, the twist of one wave being proportional to the field strength of the other.

1. Introduction

Consider the interaction of two electromagnetic waves in the absence of any other field. In Euclidean space Maxwell's equations are linear and therefore solutions can be simply superposed. That is, in the linear theory, electromagnetic waves are able to pass through each other without any interaction. In the general theory of relativity however, it is assumed that space–time is Riemannian and the coupled Einstein–Maxwell equations are highly nonlinear. That is, in general relativity there is a nonlinear interaction between electromagnetic waves. One effect of this interaction is that when two electromagnetic waves pass through each other they mutually focus each other. The foci appear as singularities in the space–time. This property is confirmed by a number of exact solutions of the Einstein–Maxwell equations (Bell and Szekeres 1974, Griffiths 1976a, b). However it has been shown (Griffiths 1976a) that these solutions are unstable with respect to small perturbations in the twist of a principal null congruence. It has also been shown that a stable solution exists in which two constant electromagnetic waves pass through each other without contraction, but with twist proportional to the magnitude of the opposing electromagnetic field. An exact solution describing this stable situation is given here.

2. The field equations

The Newman–Penrose formalism is used here. Since this formalism is now widely known it need not be re-defined, and equations may be quoted from the original papers (Newman and Penrose 1962, 1963) using the prefix NP. However, the following geometrical interpretation of some of the spin coefficients is referred to. If $\kappa = 0$ then l_μ is tangent to a geodesic null congruence with contraction, twist and shear proportional to Re ρ, Im ρ and $|\sigma|$ respectively. In the null congruence defined by the vector n_μ the spin coefficients $-\nu$, $-\mu$, $-\lambda$ correspond to κ, ρ, σ respectively.

The tetrad vectors may be chosen such that the null vectors l_μ and n_μ are parallel to the propagation vectors of the two electromagnetic waves. This is equivalent to making a Lorentz transformation to a frame of reference in which the two waves propagate in exactly opposite spatial directions. In the interaction region therefore, the electromagnetic field has two distinct principal null vectors l_μ and n_μ and Φ_1 is zero.

It is well known (Kundt 1961) that a null electromagnetic wave in otherwise empty space follows a shear-free null geodesic congruence, and that if its expansion is zero its twist must also be zero. These conclusions do not apply in the presence of the second electromagnetic wave. However, the exact solution given here in fact describes a restricted case in which both waves continue to follow shear-free null geodesics. Accordingly the assumption is made that

$$\kappa = \sigma = \nu = \lambda = 0. \tag{2.1}$$

Under this restriction Maxwell's equations remain linear in form, becoming from (NPA1)

$$\begin{aligned} &D\Phi_2 = (\rho - 2\epsilon)\Phi_2, &&\delta\Phi_2 = (\tau - 2\beta)\Phi_2, \\ &\Delta\Phi_0 = -(\mu - 2\gamma)\Phi_0, &&\bar{\delta}\Phi_0 = -(\pi - 2\alpha)\Phi_0. \end{aligned} \tag{2.2}$$

It can thus be seen that there is no interaction between the waves through Maxwell's equations. In this case therefore the interaction is purely gravitational.

The gravitational field equations (NP 4·2*a*, *n*) indicate that ρ and μ cannot be zero. When the twist is zero, ρ and μ are real and the waves mutually focus each other. However, the phase portrait of the real and imaginary parts of these equations indicates that the twist-free solutions are unstable, but that a stable solution occurs at the critical point given by

$$\rho + \bar{\rho} = 0, \qquad \mu + \bar{\mu} = 0, \tag{2.3}$$

$$\rho^2 + \Phi_{00} = 0, \qquad \mu^2 + \Phi_{22} = 0. \tag{2.4}$$

In this case the contraction of each congruence is zero, and so singularities caused by focusing do not occur. The conditions (2.4) imply that in this case the twist of each congruence is proportional to the magnitude of the electromagnetic field propagating in the opposite direction.

In seeking an exact solution of the type described equations (2.1) and (2.3) are introduced as assumptions. It is also possible to assume that

$$\tau + \bar{\pi} = 0. \tag{2.5}$$

This makes it possible to use the freedom in the choice of tetrad to put

$$\epsilon = \gamma = 0.$$

This is equivalent to assuming that there are affine parameters associated simultaneously with both tangent vectors l_μ and n_μ. The remaining tetrad freedom can now be used to put

$$\alpha + \bar{\beta} = 0. \tag{2.6}$$

Now equations (2.2) and (2.4) require that

$$\Delta\rho = 0, \qquad D\mu = 0$$

and using the field equations that

$$\delta\Phi_0 = (5\tau - 2\bar{\alpha})\Phi_0, \qquad \bar{\delta}\Phi_2 = (5\bar{\tau} - 2\alpha)\Phi_2$$

$$D\Phi_{00} = 0, \qquad \Delta\Phi_{22} = 0.$$

The components of the Weyl tensor are obtained from (NP4.2*b*, *d*, *e*, *f*, *o*, *r*, *j*) as

$$\Psi_0 = 0, \qquad \Psi_1 = -\rho\tau, \qquad \Psi_2 = \tau\bar{\tau}, \qquad \Psi_3 = \mu\bar{\tau}, \qquad \Psi_4 = 0.$$

The remaining field equations become

$$D\rho = 0$$
$$D\tau = -\rho\tau$$
$$D\alpha = \rho(\alpha - \bar{\tau})$$
$$\Delta\mu = 0$$
$$\Delta\tau = -\mu\tau$$
$$\Delta\alpha = \mu(\alpha - \bar{\tau})$$
$$\delta\rho = 3\rho\tau \tag{2.7}$$
$$\delta\mu = 3\mu\tau \tag{2.8}$$
$$\delta\tau = \tau^2 - 2\tau\bar{\alpha} + \Phi_{02}$$
$$\bar{\delta}\tau = -\rho\mu + 2\tau(\bar{\tau} + \alpha)$$
$$\delta\alpha + \overline{\delta\alpha} = \rho\mu + 4\alpha\bar{\alpha} - \tau\bar{\tau}.$$

The Bianchi identities are now automatically satisfied provided

$$D\Phi_0 = -\rho\Phi_0, \qquad \Delta\Phi_2 = \mu\Phi_2.$$

It may also be seen that the equations

$$\delta\Phi_2 = (\tau + 2\bar{\alpha})\Phi_2 \qquad \text{and} \qquad \bar{\delta}\Phi_2 = (5\bar{\tau} - 2\alpha)\Phi_2$$

are integrable provided

$$\rho\mu = \tau\bar{\tau}$$

which in turn implies that

$$\Phi_{02} = 4\tau^2.$$

The remaining integrability conditions for Φ_0 and Φ_2 are now automatically satisfied.

3. A new class of exact solutions

Under the assumptions introduced in the previous section it is possible to choose simultaneously affine parameters along the two principal directions. Labelling coordinates $x^1 = u$, $x^2 = v$, $x^3 = x$, $x^4 = y$, u and v can be chosen such that

$$l^\mu = \delta_2^\mu, \qquad n^\mu = \delta_1^\mu.$$

Following the method of Robinson and Robinson (1969) and Talbot (1969) in this case, the coordinates x and y may be chosen such that

$$m^\mu = G(Q\,\delta_1^\mu + W\delta_2^\mu + \delta_3^\mu + \mathrm{i}\,\delta_4^\mu)$$

where G, Q and W are complex functions of the coordinates. The metric tensor is now given by

$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\mu$$

where

$$l_\mu = \delta_\mu^1 - (\mathrm{Re}\; Q)\,\delta_\mu^3 - (\mathrm{Im}\; Q)\,\delta_\mu^4$$

$$n_\mu = \delta_\mu^2 - (\mathrm{Re}\; W)\,\delta_\mu^3 - (\mathrm{Im}\; W)\,\delta_\mu^4$$

$$m_\mu = -(2\bar{G})^{-1}(\delta_\mu^3 + \mathrm{i}\,\delta_\mu^4).$$

The differential operators are now

$$\mathrm{D} = \frac{\partial}{\partial v}, \qquad \Delta = \frac{\partial}{\partial u}, \qquad \delta = G\left(Q\frac{\partial}{\partial u} + W\frac{\partial}{\partial v} + \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right).$$

With the above restrictions the commutation relations become

$$(\Delta\mathrm{D} - \mathrm{D}\Delta)\phi = 0$$

$$(\delta\mathrm{D} - \mathrm{D}\delta)\phi = \tau\mathrm{D}\phi + \rho\,\delta\phi$$

$$(\delta\Delta - \Delta\delta)\phi = \tau\,\Delta\phi + \mu\,\delta\phi$$

$$(\bar{\delta}\,\delta - \delta\,\bar{\delta})\phi = -2\mu D\phi - 2\rho\,\Delta\phi - 2\bar{\alpha}\,\bar{\delta}\phi + 2\alpha\,\delta\phi$$

from which the metric equations are obtained as follows:

$$\mathrm{D}G = -\rho G \qquad \Delta G = -\mu G$$

$$\mathrm{D}Q = 0 \qquad G\Delta Q = -\tau$$

$$G\mathrm{D}W = -\tau \qquad \Delta W = 0$$

$$\bar{\delta}G - \delta\bar{G} = 2\alpha G - 2\bar{\alpha}\bar{G}$$

$$\bar{\delta}(GQ) - \delta(\overline{GQ}) = -2\rho + 2\alpha GQ - 2\overline{\alpha GQ}$$

$$\bar{\delta}(GW) - \delta(\overline{GW}) = -2\mu + 2\alpha GW - 2\overline{\alpha GW}.$$

The equations involving D and Δ can be immediately integrated to give

$$\rho = \rho(x, y)$$

$$\mu = \mu(x, y)$$

$$\tau = \tau^0\,\mathrm{e}^{-\mu u}\,\mathrm{e}^{-\rho v}$$

$$\alpha = (\alpha^0 - \mu u\bar{\tau}^0 - \rho v\bar{\tau}^0)\,\mathrm{e}^{\mu u}\,\mathrm{e}^{\rho v}$$

$$\Phi_0 = \Phi_0^0\,\mathrm{e}^{-\mu u}\,\mathrm{e}^{-\rho v}$$

$$\Phi_2 = \Phi_2^0\,\mathrm{e}^{\mu u}\,\mathrm{e}^{\rho v}$$

$$G = G^0\,\mathrm{e}^{-\mu u}\,\mathrm{e}^{-\rho v}$$

$$Q=(Q^0-\tau^0 u)/G^0$$
$$W=(W^0-\tau^0 v)/G^0$$

where quantities with a superscript zero are 'constants of integration' independent of u and v. It is now convenient to define the operator

$$\nabla=\frac{\partial}{\partial x}+\mathrm{i}\frac{\partial}{\partial y}.$$

In this notation

$$\delta=GQ\,\Delta+GWD+G\nabla.$$

Using this operator the remaining field and metric equations can be used to give the arbitrary 'constants' of integration τ^0, α^0, Φ_0^0, Φ_2^0, G^0, Q^0, W^0 in terms of ρ and μ. Equations (2.7) and (2.8) imply that

$$\tau^0=\frac{G^0}{3\rho}\nabla\rho, \qquad \text{and} \qquad \frac{1}{\rho}\nabla\rho=\frac{1}{\mu}\nabla\mu.$$

The last condition is a restriction on the symmetry of the two waves and is a consequence of the assumptions (2.5) and (2.6). It is now possible to scale the tetrad such that

$$\mu=-\rho$$

and hence that

$$\Phi_0=2\tau, \qquad \Phi_2=2\bar{\tau}.$$

The remaining equations now give

$$G^0\nabla\tau^0=(5\tau^0-2\bar{\alpha}^0-\rho Q^0+\rho W^0)\tau^0$$
$$\bar{G}^0\bar{\nabla}\tau^0=(\bar{\tau}^0+2\alpha^0-\rho\bar{Q}^0+\rho\bar{W}^0)\tau^0$$
$$G^0\nabla\alpha^0=\bar{G}^0\bar{\nabla}\bar{\alpha}^0=4\alpha^0\bar{\alpha}^0+\rho(Q^0-W^0)(\alpha^0-\bar{\tau}^0)-\rho(\bar{Q}^0-\bar{W}^0)(\bar{\alpha}^0-\tau^0)$$
$$\bar{G}^0\bar{\nabla}G^0-G^0\nabla\bar{G}^0=(2\alpha^0-\rho\bar{Q}^0+\rho\bar{W}^0)G^0-(2\bar{\alpha}^0+\rho Q^0-\rho W^0)\bar{G}^0$$
$$\bar{G}^0\bar{\nabla}Q^0-G^0\nabla\bar{Q}^0=-2\rho-(\bar{\tau}^0-2\alpha^0+\rho\bar{Q}^0-\rho\bar{W}^0)Q^0+(\tau^0-2\bar{\alpha}^0-\rho Q^0+\rho W^0)\bar{Q}^0$$
$$\bar{G}^0\bar{\nabla}W^0-G^0\nabla\bar{W}^0=2\rho-(\bar{\tau}^0-2\alpha^0+\rho\bar{Q}^0-\rho\bar{W}^0)W^0+(\tau^0-2\bar{\alpha}^0-\rho Q^0+\rho W^0)\bar{W}^0.$$

Solutions of these non-radial equations give a new class of exact solutions of the Einstein–Maxwell equations.

4. A particular example

In this section a particular solution of the non-radial equations is given. It has been found convenient to introduce cylindrical polar coordinates, so that

$$x+\mathrm{i}y=r\,\mathrm{e}^{\mathrm{i}\theta}.$$

The particular solution is

$$\tau^0 = k/(r+a)^3$$
$$\rho = \mathrm{i}\tau^0$$
$$\alpha^0 = \frac{k}{(r+a)^3}\left(2 - \frac{b}{r+a}\right)$$
$$G^0 = -\frac{k}{(r+a)^2}\,\mathrm{e}^{-\mathrm{i}\theta}$$
$$Q^0 = \mathrm{i}\left(1 - \frac{b}{r+a}\right)$$
$$W^0 = -Q^0$$

where a, b and k are arbitrary real constants. To summarize, the metric is given by

$$\mathrm{d}s^2 = -\frac{(r+a)^4}{2k^2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2) + 2l_\mu\,\mathrm{d}x^\mu n_\nu\,\mathrm{d}x^\nu$$

where

$$l_\mu\,\mathrm{d}x^\mu = \mathrm{d}u - \frac{u}{r+a}\,\mathrm{d}r + \frac{1}{k}(r+a)(r+a-b)r\,\mathrm{d}\theta$$
$$n_\mu\,\mathrm{d}x^\mu = \mathrm{d}v - \frac{v}{r+a}\,\mathrm{d}r - \frac{1}{k}(r+a)(r+a-b)r\,\mathrm{d}\theta$$

and the electromagnetic field is defined by

$$\Phi_0 = \frac{2k}{(r+a)^3}\exp\left(\frac{\mathrm{i}k(u-v)}{(r+a)^3}\right), \qquad \Phi_2 = \frac{2k}{(r+a)^3}\exp\left(\frac{\mathrm{i}k(v-u)}{(r+a)^3}\right).$$

5. Conclusions

In the above a new class of exact solutions of the Einstein–Maxwell field equations has been given. They correspond to a particular class of solutions for which the electromagnetic field has two principal null directions and may be interpreted as describing two electromagnetic waves propagating in different directions. The waves follow shear-free null geodesics which are affinely parametrized and, since the condition has been made that the twist of the rays is equal, with this choice of units, to the field strength of the opposing wave, the contraction of the waves is zero and there is no focusing effect.

An interesting feature of this solution is that it is stable with respect to small perturbations in contraction and twist. It is also of interest to notice that the solution obtained here differs considerably from a solution given elsewhere (Griffiths 1975) describing two gravitational waves twisting through each other with zero contraction, in spite of very similar initial assumptions.

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